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The *R*-matrix approach to integrable systems on time scales

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Abstract
A general unifying framework for integrable soliton-like systems on time scales is introduced. The *R*-matrix formalism is applied to the algebra of δ-differential operators in terms of which one can construct an infinite hierarchy of commuting vector fields. The theory is illustrated by two infinite-field integrable hierarchies on time scales which are Δ-differential counterparts of KP and mKP. The difference counterparts of AKNS and Kaup–Broer soliton systems are constructed as related finite-field restrictions.

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1. Introduction

Integrable systems are widely investigated in $(1+1)$ dimensions, where one of the dimensions stands for the time evolution variable and the other one stands for the space variable. The space variable is usually considered on continuous intervals, or both on integer values and on $\mathbb{R}$ [1] or on $\mathbb{K}_q$ intervals [2, 3]. In order to embed the study of integrable systems into a more general unifying framework, one of the possible approaches is to construct the integrable systems on time scales. Here the space variable is considered on any time scale where $\mathbb{R}$, $\hbar\mathbb{Z}$, $\mathbb{K}_q$ are special cases. The first step in this direction was taken in [4], where the Gelfand–Dickey approach [5, 6] was extended in order to construct integrable nonlinear evolutionary equations on any time scale. Another unifying approach is to formulate different types of discrete dynamics on $\mathbb{R}$. Some contribution in this direction was made recently in [7].

The main goal of this work is to present a theory for the systematic construction of $(1+1)$-dimensional integrable Δ-differential systems on time scales in the frame of the *R*-matrix formalism. By an integrable system, we mean such a system which has an infinite-hierarchy of mutually commuting symmetries. The *R*-matrix formalism is one of the most effective and systematic methods of constructing integrable systems [8, 9]. This formalism originated from the pioneering article [5] by Gelfand and Dickey, who constructed the soliton systems of KdV type. The crucial point of the R-matrix formalism is that the construction of integrable systems proceeds from the Lax equations on appropriate Lie algebras [8, 9]. The simplest R-matrices can be constructed by a decomposition of a given Lie algebra into two Lie subalgebras. We refer to [1, 6, 9] for abstract formalism of classical R-matrices on Lie algebras.

This paper is organized as follows: in the following section, we give a brief review of the time scale calculus. In the third section, we define the δ-differentiation operator and formulate the Leibniz rule for this operator. We introduce the Lie algebra as an algebra of δ-differential operators equipped with the commutator, decompose it into two Lie subalgebras and construct the simplest R-matrix on this algebra. We present the appropriate Lax operators for infinite-field cases and the admissible finite-field restrictions generating consistent Lax hierarchies. In the $\mathbb{T} = \mathbb{R}$ case, or in the continuous limit of some special time scales, we observe that the algebra of δ-differential operators turns out to be the algebra of pseudo-differential operators. Next, we formulate and prove the property of the algebra of δ-differential operators. This property allows us to obtain natural constraints which are fulfilled by finite-field restrictions. Therefore, the source of the constraints, obtained in the Burgers equations and KdV hierarchy on time scales in [4], is established. We end up this section with the construction of the recursion operators by means of the method presented in [10]. In the fourth section, we illustrate two infinite-field integrable hierarchies on time scales which are Δ-differential counterparts of Kadomtsev–Petviashvili (KP) and modified Kadomtsev–Petviashvili (mKP) hierarchies. In the last section, we present finite-field restrictions which are difference counterparts of Ablowitz–Kaup–Newell–Segur (AKNS) and Kaup–Broer (KB) hierarchies with their recursion operators.

2. Preliminaries

In this section, we give a brief introduction to the concept of time scale. We refer to [11, 12] for the basic definitions and general theory of time scale. What we mean by a time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers. The time scale calculus was introduced by Aulbach and Hilger [13, 14] in order to unify all possible intervals on the real line $\mathbb{R}$, such as continuous (whole) $\mathbb{R}$, discrete $\mathbb{Z}$ and q-discrete $\mathbb{K}_q$ ($\mathbb{K}_q = q^{\mathbb{Z}} \cup \{0\} \equiv \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, where $q \neq 1$ is a fixed real number) intervals. For the definition of the derivative in time scales, we use *forward* and *backward jump operators* which are defined as follows.

Definition 2.1. *For $x \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(x) = \inf\{y \in \mathbb{T} : y > x\}, \tag{2.1}$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(x) = \sup\{y \in \mathbb{T} : y < x\}. \tag{2.2}$$

We set in addition $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, and $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$.

The jump operators σ and ρ allow the classification of points in a time scale in the following way: x is called right dense, right scattered, left dense, left scattered, dense and isolated if $\sigma(x) = x, \sigma(x) > x, \rho(x) = x, \rho(x) < x, \sigma(x) = \rho(x) = x$ and $\rho(x) < x < \sigma(x)$, respectively. Moreover, we define the graininess functions $\mu, \nu : \mathbb{T} \to [0, \infty)$ as follows:

$$\mu(x) = \sigma(x) - x, \qquad \nu(x) = x - \rho(x), \qquad \textit{for all} \quad x \in \mathbb{T}. \tag{2.3}$$

In the literature, $\mathbb{T}^\kappa$ denotes a set consisting of $\mathbb{T}$ except for a possible left-scattered maximal point while $\mathbb{T}_\kappa$ stands for a set of points of $\mathbb{T}$ except for a possible right-scattered minimal point.

Definition 2.2. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function on a time scale $\mathbb{T}$. For $x \in \mathbb{T}^\kappa$, the delta derivative of f, denoted by Δf, is defined as*

$$\Delta f(x) = \lim_{s\to x} \frac{f(\sigma(x)) - f(s)}{\sigma(x) - s}, \qquad s \in \mathbb{T}, \tag{2.4}$$

while for $x \in \mathbb{T}_\kappa$, ∇-derivative of f, denoted by ∇f, is defined as

$$\nabla f(x) = \lim_{s\to x} \frac{f(s) - f(\rho(x))}{s - \rho(x)}, \qquad s \in \mathbb{T}, \tag{2.5}$$

provided that the limits exist. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be Δ-smooth (∇-smooth) if it is infinitely Δ-differentiable (∇-differentiable).

Remark 2.3. Let $f : \mathbb{T} \to \mathbb{R}$ be Δ-differentiable on $\mathbb{T}^\kappa$. If x is right-scattered, then the definition (2.4) turns out to be

$$\Delta f(x) = \frac{f(\sigma(x)) - f(x)}{\mu(x)},$$

while if x is right-dense, (2.4) implies that

$$\Delta f(x) = \lim_{s\to x} \frac{f(x) - f(s)}{x - s}, \qquad s \in \mathbb{T}.$$

Similarly, let $f : \mathbb{T} \to \mathbb{R}$ be ∇-differentiable on $\mathbb{T}_\kappa$. If x is left-scattered, then the definition (2.5) turns out to be

$$\nabla f(x) = \frac{f(x) - f(\rho(x))}{\nu(x)},$$

while if x is left-dense, (2.5) yields

$$\nabla f(x) = \lim_{s\to x} \frac{f(x) - f(s)}{x - s}, \qquad s \in \mathbb{T}.$$

In order to be more precise, we present Δ and ∇ derivatives for some special time scales. If $\mathbb{T} = \mathbb{R}$, then Δ- and ∇-derivatives become ordinary derivatives, i.e.

$$\Delta f(x) = \nabla f(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}.$$

If $\mathbb{T} = \hbar\mathbb{Z}$, then

$$\Delta f(x) = \frac{f(x+\hbar) - f(x)}{\hbar} \qquad \text{and} \qquad \nabla f(x) = \frac{f(x) - f(x-\hbar)}{\hbar}.$$

If $\mathbb{T} = \mathbb{K}_q$, then

$$\Delta f(x) = \frac{f(qx) - f(x)}{(q-1)x} \qquad \text{and} \qquad \nabla f(x) = \frac{f(x) - f(q^{-1}x)}{(1-q^{-1})x},$$

for all $x \neq 0$, and

$$\Delta f(0) = \nabla f(0) = \lim_{s\to 0} \frac{f(s) - f(0)}{s}, \qquad s \in \mathbb{K}_q,$$

provided that this limit exists.

As an important property of Δ-differentiation on $\mathbb{T}$, we give the product rule. If $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable functions at $x \in \mathbb{T}^{\kappa}$, then their product is also Δ-differentiable and the following Lebniz-like rule holds:

$$\Delta(fg)(x) = g(x)\Delta f(x) + f(\sigma(x))\Delta g(x)$$
$$= f(x)\Delta g(x) + g(\sigma(x))\Delta f(x). \tag{2.6}$$

Besides, if f is a Δ-smooth function, then

$$f(\sigma(x)) = f(x) + \mu(x)\Delta f(x). \tag{2.7}$$

If $x \in \mathbb{T}$ is right-dense, then $\mu(x) = 0$ and the relation (2.7) is trivial.

Definition 2.4. *A time scale $\mathbb{T}$ is regular if both of the following two conditions are satisfied:*

(i) $\sigma(\rho(x)) = x$ *for all* $x \in \mathbb{T}$,
(ii) $\rho(\sigma(x)) = x$ *for all* $x \in \mathbb{T}$.

Set $x_* = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$, and set $x_* = -\infty$ otherwise. Also set $x^* = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, and set $x^* = \infty$ otherwise.

Proposition 2.5 [4]**.** *A time scale is regular if and only if the following two conditions hold:*

(i) the point $x_* = \min \mathbb{T}$ *is right-dense and the point* $x^* = \max \mathbb{T}$ *is left-dense;*
(ii) each point of $\mathbb{T}\backslash\{x_*, x^*\}$ *is either two-sided dense or two-sided scattered.*

In particular $\mathbb{R}, \hbar\mathbb{Z}$ $(\hbar \neq 0)$ and $\mathbb{K}_q$ are regular time scales, as are $[0, 1]$ and $[-1, 0] \cup \{1/k : k \in \mathbb{N}\} \cup \{k/(k+1) : k \in \mathbb{N}\} \cup [1, 2]$.

Throughout this work, let $\mathbb{T}$ be a regular time scale. By Δ, we denote the delta-differentiation operator which assigns each Δ-differentiable function $f : \mathbb{T} \to \mathbb{R}$ to its delta-derivative $\Delta(f)$, defined by

$$[\Delta(f)](x) = \Delta f(x), \qquad \text{for} \quad x \in \mathbb{T}^{\kappa}. \tag{2.8}$$

The *shift operator* E is defined by the formula

$$(Ef)(x) = f(\sigma(x)), \qquad x \in \mathbb{T}. \tag{2.9}$$

The inverse E^{-1} is defined by

$$(E^{-1}f)(x) = f(\sigma^{-1}(x)) = f(\rho(x)), \tag{2.10}$$

for all $x \in \mathbb{T}$. Note that E^{-1} exists only in the case of regular time scales and that in general E and E^{-1} do not commute with Δ and ∇ operators.

Proposition 2.6 [15]**.** *Let $\mathbb{T}$ be a regular time scale.*

(i) If $f : \mathbb{T} \to \mathbb{R}$ *is a* Δ*-smooth function on* $\mathbb{T}^{\kappa}$*, then* f *is* ∇*-smooth and for all* $x \in \mathbb{T}_{\kappa}$,

$$\nabla f(x) = E^{-1}\Delta f(x). \tag{2.11}$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ *is a* ∇*-smooth function on* $\mathbb{T}_{\kappa}$*, then* f *is* Δ*-smooth and for all* $x \in \mathbb{T}^{\kappa}$,

$$\Delta f(x) = E\nabla f(x). \tag{2.12}$$

Thus the properties of Δ- and ∇-smoothness for functions on regular time scales are equivalent.

In some special cases, by properly introducing the deformation parameter, it is possible to consider a continuous limit of a time scale. For instance, the continuous limit of $\hbar\mathbb{Z}$ is the whole real line $\mathbb{R}$, i.e.

$$\mathbb{T} = \hbar\mathbb{Z} \xrightarrow{\hbar\to 0} \mathbb{T} = \mathbb{R}; \tag{2.13}$$

and the continuous limit of $\mathbb{K}_q$ is the closed half-line $\mathbb{R}_+ \cup \{0\}$, thus

$$\mathbb{T} = \mathbb{K}_q \xrightarrow{\hbar\to 1} \mathbb{T} = \mathbb{R}_+ \cup 0. \tag{2.14}$$

For more about the calculus on time scales we refer the readers to [11, 12].

3. Algebra of δ-differential operators

3.1. Basic notions

In this section, we deal with the algebra of δ-differential operators defined on a regular time scale $\mathbb{T}$. We denote the delta-differentiation operator by δ instead of Δ, for convenience in the operational relations. The operator δf which is a composition of δ and f, where $f : \mathbb{T} \to \mathbb{R}$, is introduced as follows:

$$\delta f := \Delta f + E(f)\delta, \qquad \forall\, f. \tag{3.1}$$

Note that, the definition (3.1) is consistent with the Lebniz-like rule on time scales (2.6).

Theorem 3.1. *The Leibniz rule on time scales for the operator δ is given as follows.*

(i) For $n \geqslant 0$:

$$\delta^n f = \sum_{k=0}^{n} \sum_{i_1+i_2+\cdots+i_{k+1}=n-k} (\Delta^{i_{k+1}} E \Delta^{i_k} E \ldots \Delta^{i_2} E \Delta^{i_1}) f \delta^k, \tag{3.2}$$

where $i_\gamma \geqslant 0$ for all $\gamma = 1, 2, \ldots, k+1$. Here the formula includes all possible strings containing $n-k$ times Δ and k times E.

(ii) For $n < 0$:

$$\delta^n f = \sum_{k=-n}^{\infty} \sum_{i_1+i_2+\cdots+i_{k+n+1}=k} (-1)^{k+n} (E^{-i_{k+n+1}} \Delta E^{-i_{k+n}} \Delta \ldots E^{-i_2} \Delta E^{-i_1}) f \delta^{-k}, \tag{3.3}$$

where $i_\gamma > 0$ for all $\gamma = 1, 2, \ldots, k+n+1 > 0$. Here the formula includes strings of the length $2k+2n+1$.

The above theorem is a straightforward consequence of definition (3.1). Note that $\delta^{-1} f$ has the form of the formal series

$$\delta^{-1} f = \sum_{k=0}^{\infty} (-1)^k ((E^{-1}\Delta)^k E^{-1}) f \delta^{-k-1}, \tag{3.4}$$

which was previously given in [4], in terms of ∇. Thus (3.3) is the appropriate generalization of (3.4).

3.2. Classical R-matrix formalism

In order to construct integrable hierarchies of mutually commuting vector fields on time scales, we deal with a systematic method, so-called *the classical R-matrix formalism* [1, 6, 9], presented in the following scheme.

Let $\mathcal{G}$ be an algebra, with some associative multiplication operation, over a commutative field $\mathbb{K}$ of complex or real numbers, based on an additional bilinear product given by a Lie bracket $[\cdot,\cdot]:\mathcal{G}\to\mathcal{G}$, which is skew-symmetric and satisfies the Jacobi identity.

Definition 3.2. *A linear map $R:\mathcal{G}\to\mathcal{G}$ such that the bracket*

$$[a,b]_R := [Ra,b]+[a,Rb] \tag{3.5}$$

is a second Lie bracket on $\mathcal{G}$, is called the classical R-matrix.

Skew-symmetry of (3.5) is obvious. When one checks the Jacobi identity of (3.5), it can be clearly deduced that a sufficient condition for R to be a classical R-matrix is

$$[Ra,Rb]-R[a,b]_R+\alpha[a,b]=0, \tag{3.6}$$

where $\alpha\in\mathbb{K}$, called the *Yang–Baxter equation* YB(α). There are only two relevant cases of YB(α), namely $\alpha\neq0$ and $\alpha=0$, as Yang–Baxter equations for $\alpha\neq0$ are equivalent up to reparametrization.

Additionally, assume that the Lie bracket is a derivation of multiplication in $\mathcal{G}$, i.e. the relation

$$[a,bc]=b[a,c]+[a,b]c \qquad a,b,c\in\mathcal{G} \tag{3.7}$$

holds. If the Lie bracket is given by the commutator, i.e. $[a,b]=ab-bc$, the condition (3.7) is satisfied automatically, since $\mathcal{G}$ is associative.

Proposition 3.3. *Let $\mathcal{G}$ be a Lie algebra fulfilling all the above assumptions and R be the classical R-matrix satisfying the Yang–Baxter equation, YB(α). Then the power functions L^n on $\mathcal{G}$, $L\in\mathcal{G}$ and $n\in\mathbb{Z}_+$, generate the so-called Lax hierarchy*

$$\frac{\mathrm{d}L}{\mathrm{d}t_n}=[R(L^n),L], \tag{3.8}$$

of pairwise commuting vector fields on $\mathcal{G}$. Here, t_n's are related evolution parameters. We additionally assume that R commutes with derivatives with respect to these evolution parameters.

Proof. It is clear that the power functions on $\mathcal{G}$ are well defined. Then

$$\begin{aligned}(L_{t_m})_{t_n}-(L_{t_n})_{t_m}&=[RL^m,L]_{t_n}-[RL^n,L]_{t_m}\\&=[(RL^m)_{t_n}-(RL^n)_{t_m},L]+[RL^m,[RL^n,L]]-[RL^n,[RL^m,L]]\\&=[(RL^m)_{t_n}-(RL^n)_{t_m}+[RL^m,RL^n],L].\end{aligned}$$

Hence, the vector fields (3.8) mutually commute if the so-called *zero-curvature* (or Zakharov–Shabat) *equations*

$$(RL^m)_{t_n}-(RL^n)_{t_m}+[RL^m,RL^n]=0$$

are satisfied. From (3.8) and by the Leibniz rule (3.7) we have that $(L^m)_{t_n}=[RL^n,L^m]$. Using the Yang–Baxter equation for R and the fact that R commutes with ∂_{t_n}, we deduce

$$\begin{aligned}&R(L^m)_{t_n}-R(L^n)_{t_m}+[RL^m,RL^n]\\&\quad=R[RL^n,L^m]-R[RL^m,L^n]+[RL^m,RL^n]\\&\quad=[RL^m,RL^n]-R[L^m,L^n]_R=-\alpha[L^m,L^n]=0.\end{aligned}$$

Hence, the vector fields commute pairwise. □

In practice the powers of Lax operators in (3.8) are fractional. Note that, the Yang–Baxter equation is a sufficient condition for mutual commutation of vector fields (3.8), but not necessary. Thus choosing an algebra $\mathcal{G}$ properly, the Lax hierarchy yields abstract integrable systems. In practice, the element L of $\mathcal{G}$ must be appropriately chosen, in such a way that the evolution systems (3.8) are consistent on the subspace of $\mathcal{G}$.

3.3. Classical R-matrix on time scales

We introduce the algebra $\mathcal{G}$ as an algebra of the formal Laurent series of (pseudo-) δ-differential operators equipped with the commutator, and define its decomposition such as

$$\mathcal{G} = \mathcal{G}_{\geqslant k} \oplus \mathcal{G}_{<k} = \left\{ \sum_{i\geqslant k} u_i(x)\delta^i \right\} \oplus \left\{ \sum_{i<k} u_i(x)\delta^i \right\}, \tag{3.9}$$

where $u_i : \mathbb{T} \to \mathbb{K}$ are Δ-smooth functions. The subspaces $\mathcal{G}_{\geqslant k}, \mathcal{G}_{<k}$ are closed Lie subalgebras of $\mathcal{G}$ only if $k = 0, 1$. Thus, we define the classical R-matrix in the following form:

$$R := \tfrac{1}{2}(P_{\geqslant k} - P_{<k}) \qquad k = 0, 1, \tag{3.10}$$

where $P_{\geqslant k}$ and $P_{<k}$ are the projections onto $\mathcal{G}_{\geqslant k}$ and $\mathcal{G}_{<k}$, respectively. Since the classical R-matrices (3.10) are defined through the projections onto Lie subalgebras, they satisfy the Yang–Baxter equation (3.6) for $\alpha = \frac{1}{4}$.

Let $L \in \mathcal{G}$ be given in the form

$$L = u_N\delta^N + u_{N-1}\delta^{N-1} + \cdots + u_1\delta + u_0 + u_{-1}\delta^{-1} + \cdots, \tag{3.11}$$

where u_i are dynamical fields depending additionally on the evolution parameters t_n. Thus, the Lax hierarchy (3.8), based on (3.10) and in general generated by fractional powers of L, turns out to be

$$\frac{\mathrm{d}L}{\mathrm{d}t_n} = [(L^{\frac{n}{N}})_{\geqslant k}, L] = -[(L^{\frac{n}{N}})_{<k}, L] \qquad k = 0, 1 \quad n \in \mathbb{Z}_+. \tag{3.12}$$

Proposition 3.3 implies that the hierarchy (3.12) is the infinite hierarchy of mutually commuting vector fields and represents (1 + 1)-dimensional integrable Δ-differential systems on a time scale $\mathbb{T}$, including the time variables t_n and space variable $x \in \mathbb{T}$.

Analyzing (3.12) for L given by (3.11), in the case of $k = 0$, one finds that $(u_N)_t = 0$ and $(u_{N-1})_t = \mu(\ldots)$ (see also remark 4.1). Similarly for $k = 1$, we have $(u_N)_t = \mu(\ldots)$ (see also remark 4.2). Hence, the appropriate Lax operators, yielding consistent Lax hierarchies (3.12), are in the following form:

$$k = 0: \qquad L = c_N\delta^N + \tilde{u}_{N-1}\delta^{N-1} + \cdots + u_1\delta^1 + u_0 + u_{-1}\delta^{-1} + \cdots \tag{3.13}$$

$$k = 1: \qquad L = \tilde{u}_N\delta^N + u_{N-1}\delta^{N-1} + \cdots + u_1\delta^1 + u_0 + u_{-1}\delta^{-1} + \cdots, \tag{3.14}$$

where c_N is a time-independent field and fields $\tilde{u}_{N-1}, \tilde{u}_N$ are time-independent for dense $x \in \mathbb{T}$, as at these points $\mu = 0$. This is the reason why they are distinguished by a tilde mark.

Nevertheless, we are interested in finite-field integrable systems on time scales. Thus, in order to work with a finite number of fields, we should impose some restrictions on (3.13) and (3.14) in such a way that the commutator on the right-hand side of the Lax equation (3.12) does not produce terms not contained in the left-hand side of the Lax equation. To be more precise, the left- and right-hand side of (3.12) span the same subspace of $\mathcal{G}$. From this purpose, in the case of $k = 0$, one finds the general admissible form of the finite-field Lax operator given by

$$L = c_N\delta^N + \tilde{u}_{N-1}\delta^{N-1} + \cdots + u_1\delta + u_0 + \sum_s \psi_s\delta^{-1}\varphi_s, \tag{3.15}$$

with further restriction

$$L = c_N\delta^N + \tilde{u}_{N-1}\delta^{N-1} + \cdots + u_1\delta + u_0. \tag{3.16}$$

In the case of $k = 1$, the general admissible Lax operator has the form

$$L = \tilde{u}_N\delta^N + u_{N-1}\delta^{N-1} + \cdots + u_1\delta + u_0 + \delta^{-1}u_{-1} + \sum_s \psi_s\delta^{-1}\varphi_s, \tag{3.17}$$

and further restrictions are

$$L = \tilde{u}_N\delta^N + u_{N-1}\delta^{N-1} + \cdots + u_1\delta + u_0 + \delta^{-1}u_{-1} \tag{3.18}$$

$$L = \tilde{u}_N\delta^N + u_{N-1}\delta^{N-1} + \cdots + u_1\delta + u_0 \tag{3.19}$$

$$L = \tilde{u}_N\delta^N + u_{N-1}\delta^{N-1} + \cdots + u_1\delta. \tag{3.20}$$

In the above Lax operators c_N is a time-independent field for all $x \in \mathbb{T}$ and $\tilde{u}_{N-1}, \tilde{u}_N$ are time-independent at dense points from a time scale. We also assume that the sum $\sum_s$ is finite.

In general, for an arbitrary regular time scale $\mathbb{T}$, the Lax hierarchies (3.12) represent hierarchies of soliton-like integrable Δ-differential systems. For instance, when $\mathbb{T} = \hbar\mathbb{Z}$ or $\mathbb{K}_q$, the hierarchies (3.12) are those of lattice and q-deformed (-like) (discrete) soliton systems, respectively. In particular, for the case of $\mathbb{T} = \mathbb{R}$, i.e. the continuous time scale on the whole $\mathbb{R}$, the Lax hierarchies are those of field soliton systems. In some cases, field soliton systems can also be obtained from the continuous limit of integrable systems on time scales (see (2.13) and (2.14)).

In the continuous time scale, the algebra of δ-differential operators (3.9) turns out to be the algebra of pseudo-differential operators

$$\mathcal{G} = \mathcal{G}_{\geqslant k} \oplus \mathcal{G}_{<k} = \left\{\sum_{i\geqslant k} u_i(x)\partial^i\right\} \oplus \left\{\sum_{i<k} u_i(x)\partial^i\right\}, \tag{3.21}$$

where ∂ is such that $\partial u = \partial_x u + u\partial = u_x + u\partial$. The above decomposition is valid only if $k = 0$, 1 and 2. Thus, in the general theory of integrable systems on time scales, we loose one case in contrast to the ordinary soliton systems constructed by means of pseudo-differential operators. This follows from the fact that, for $k = 2$, (3.9) does not decompose into Lie subalgebras for an arbitrary time scale. For appropriate Lax operators, finite-field restrictions and more information about the algebra of pseudo-differential operators, we refer the reader to [1, 6, 16, 17]. Note that the fields ψ_s and φ_s in (3.15) and (3.17) are special dynamical fields in the case of the algebra of pseudo-differential operators. They are the so-called source terms, as ψ_s and φ_s are eigenfunctions and adjoint-eigenfunctions, respectively, of the Lax hierarchy (3.12) [17].

It turns out that there are constraints between dynamical fields of the admissible finite-field Lax restrictions (3.15)–(3.18) fulfilling (3.12). We give these constraints in the following theorem, which is a consequence of the property of the algebra of δ-differential operators. This property is illustrated in the following lemma:

Lemma 3.4. *Consider the equality*

$$\delta^r F = \sum_{i=0}^{r} C_i \delta^{r-i}, \qquad r > 0. \tag{3.22}$$

Then the following relation:

$$\sum_{i=0}^{r} (-\mu)^i C_i = F \tag{3.23}$$

is valid.

Proof. We make use of induction. Assume that (3.23) holds for r. Then

$$\delta^{r+1} F = \delta^r (EF)\delta + \delta^r \Delta F = \sum_{i=0}^{r} A_i \delta^{r-i+1} + \sum_{i=0}^{r} B_i \delta^{r-i} = \sum_{i=0}^{r+1} C_i \delta^{r+1-i}. \tag{3.24}$$

By the assumption we have $\sum_{i=0}^{r} (-\mu)^i A_i = EF$ and $\sum_{i=0}^{r} (-\mu)^i B_i = \Delta F$. Hence

$$\sum_{i=0}^{r+1} (-\mu)^i C_i = \sum_{i=0}^{r} (-\mu)^{(i+1)} B_i + \sum_{i=0}^{r} (-\mu)^i A_i = -\mu \Delta F + EF = F. \tag{3.25}$$

□

Let us explain the source of lemma 3.4. Consider the equality

$$A = \sum_{i \geqslant 0} a_i \delta^i = 0, \tag{3.26}$$

where the sum is finite, and A is a purely δ-differential operator. We expand A with respect to the shift operator $\mathcal{E}$: $\mathcal{E}u = E(u)\mathcal{E}$. From the relation (2.7) we have

$$\mathcal{E} = 1 + \mu\delta. \tag{3.27}$$

The equality from lemma 3.4 is trivially satisfied for dense $x \in \mathbb{T}$, since in this case $\mu = 0$. Thus, it is enough to consider remaining points in a time scale so assume that $\mu \neq 0$. Hence, from (3.27), we have the formula

$$\delta = \mu^{-1}\mathcal{E} - \mu^{-1}. \tag{3.28}$$

Thus, using (3.28) the relation (3.26) can be rewritten as

$$A = \sum_{i} a'_i \mathcal{E}^i = 0. \tag{3.29}$$

Obviously, it must hold for terms of all orders. The equality for the zero-order terms, i.e. $a'_0 = 0$, can be simply obtained by replacing δ with $-\mu^{-1}$ in (3.26). The same substitution in (3.22) allows us to find

$$(-\mu)^{-r} F = \sum_{i=0}^{r} C_i (-\mu)^{-r+i}, \tag{3.30}$$

which is equivalent to (3.23).

The above procedure can also be extended to operators A that are not purely δ-differential and contain finitely many terms with $\delta^{-1}, \delta^{-2}, \ldots$. As an illustration consider the equality

$$[A\delta^r, \psi\delta^{-1}\varphi] = \sum_{i=0}^{r-1} C_i \delta^{r-1-i} + \hat{C}_r \delta^{-1}\varphi + \psi\delta^{-1} C_r. \tag{3.31}$$

The above equality is well formulated since it follows immediately from the definition and the property of the δ operator. Replacing δ with $-\mu^{-1}$, the commutator vanishes, and we have

$$0 = \sum_{i=0}^{r-1} C_i(-\mu)^{-r+1+i} + \hat{C}_r(-\mu)\varphi + \psi(-\mu)C_r \quad \Longleftrightarrow \tag{3.32}$$

$$\sum_{i=0}^{r-1}(-\mu)^i C_i + (-\mu)^r(\hat{C}_r\varphi + \psi C_r) = 0. \tag{3.33}$$

Straightforward consequence of such a behavior of δ-differential operators is the next theorem.

Theorem 3.5

(i) The case $k = 0$. The constraint between dynamical fields of (3.15), generating Lax hierarchy (3.12), has the form

$$\begin{aligned}&(-\mu)^{N-1}\frac{\mathrm{d}\tilde{u}_{N-1}}{\mathrm{d}t_n} + \sum_{i=0}^{N-2}(-\mu)^i\frac{\mathrm{d}u_i}{\mathrm{d}t_n} - \mu\sum_s\frac{\mathrm{d}(\psi_s\varphi_s)}{\mathrm{d}t_n} = 0\\ &\Longrightarrow \quad (-\mu)^{N-1}\tilde{u}_{N-1} + \sum_{i=0}^{N-2}(-\mu)^i u_i - \mu\sum_s\psi_s\varphi_s = a_n,\end{aligned} \tag{3.34}$$

where $n \in \mathbb{Z}_+$ and a_n is a time-independent function.

(ii) The case $k = 1$. The constraint between dynamical fields of (3.17), generating (3.12), has the form

$$\begin{aligned}&(-\mu)^{N}\frac{\mathrm{d}\tilde{u}_{N}}{\mathrm{d}t_n} + \sum_{i=-1}^{N-1}(-\mu)^i\frac{\mathrm{d}u_i}{\mathrm{d}t_n} - \mu\sum_s\frac{\mathrm{d}(\psi_s\varphi_s)}{\mathrm{d}t_n} = 0\\ &\Longrightarrow \quad (-\mu)^{N}\tilde{u}_{N} + \sum_{i=-1}^{N-1}(-\mu)^i u_i - \mu\sum_s\psi_s\varphi_s = a_n,\end{aligned} \tag{3.35}$$

where $n \in \mathbb{Z}_+$ and a_n is a time-independent function.

Proof. We already know that the Lax operators (3.15) and (3.17) generate consistent Lax hierarchies (3.12). Thus, the right-hand side of (3.12) can be represented in the form of L_{t_n}. Replacing δ with $-\mu^{-1}$ in (3.12), we have

$$L_{t_n}|_{\delta=-\mu^{-1}} = [(L^n)_{\geqslant k}, L]|_{\delta=-\mu^{-1}} = 0. \tag{3.36}$$

Hence, the constraints (3.34) and (3.35) follow. □

The above theorem can be generalized to further restrictions. As a consequence, the constraints (3.34) or (3.35) with fixed common value of all a_n are valid for the whole Lax hierarchy (3.12).

3.4. Recursion operators

One of the characteristic features of integrable systems possessing infinite-hierarchy of mutually commuting symmetries is the existence of a recursion operator [1, 18]. A recursion operator of a given system is an operator of such property that when it acts on one symmetry of the system considered, it produces another symmetry. Gürses *et al* [10] presented a general and very efficient method of constructing recursion operators for Lax hierarchies. Among others, the authors illustrated the method by applying it to finite-field reductions of the KP hierarchy. In [19] the method was applied to the reductions of the modified KP hierarchy as well as to the lattice systems. Our further considerations are based on the scheme from [10] and [19].

The recursion operator Φ has the following property:

$$\Phi(L_{t_n}) = L_{t_{n+N}}, \qquad n \in \mathbb{Z}_+,$$

and hence it allows the reconstruction of the whole hierarchy (3.12) when applied to the first $(N-1)$ symmetries.

Lemma 3.6

(i) The case $k=0$. Let the Lax operator be given in the general form (3.15). Then, the recursion operator of the related Lax hierarchy can be constructed solving

$$L_{t_{n+N}} = L_{t_n}L + [R, L] \tag{3.37}$$

with the remainder in the form

$$R = a_{N-1}\delta^{N-1} + \cdots + a_0 + \sum_s a_{-1,s}\delta^{-1}\varphi_s, \tag{3.38}$$

where N is the highest order of L.

(ii) The case $k=1$. Similarly for the Lax operator (3.17), the recursion operator can be constructed from (3.37) with

$$R = a_N\delta^N + \cdots + a_0 + \sum_s a_{-1,s}\delta^{-1}\varphi_s. \tag{3.39}$$

Proof. Consider the case $k=0$. Then for (3.15) we have

$$\begin{aligned}(L^{\frac{n+N}{N}})_{\geqslant 0} &= ((L^{\frac{n}{N}})_{\geqslant 0}L)_{\geqslant 0} + ((L^{\frac{n}{N}})_{<0}L)_{\geqslant 0}\\ &= (L^{\frac{n}{N}})_{\geqslant 0}L - \sum_s [(L^{\frac{n}{N}})_{\geqslant 0}\psi_s]_0\delta^{-1}\varphi_s + ((L^{\frac{n}{N}})_{<0}L)_{\geqslant 0}\\ &= (L^{\frac{n}{N}})_{\geqslant 0}L + R,\end{aligned}$$

where $\left[\sum_i a\delta^i\right]_0 = a_0$ and R is given by (3.38). Similarly for $k=1$, we have

$$\begin{aligned}\left(L^{\frac{n+N}{N}}\right)_{\geqslant 1} &= ((L^{\frac{n}{N}})_{\geqslant 1}L)_{\geqslant 1} + ((L^{\frac{n}{N}})_{<1}L)_{\geqslant 1}\\ &= (L^{\frac{n}{N}})_{\geqslant 1}L - [(L^{\frac{n}{N}})_{\geqslant 1}L]_0 - \sum_s [(L^{\frac{n}{N}})_{\geqslant 0}\psi_s]_0\delta^{-1}\varphi_s + ((L^{\frac{n}{N}})_{<1}L)_{\geqslant 1}\\ &= (L^{\frac{n}{N}})_{\geqslant 1}L + R,\end{aligned}$$

where R has the form (3.39). Thus, in both cases (3.37) follows from (3.12). Hence we can extract the recursion operator from (3.37). □

Note that in general, recursion operators on time scales are non-local. This means that they contain non-local terms with Δ^{-1} being the formal inverse of Δ operator. However, such recursion operators acting on an appropriate domain produce only local hierarchies.

4. Infinite-field integrable systems on time scales

4.1. Δ*-differential KP,* $k = 0$*:*

Consider the following infinite-field Lax operator:

$$L = \delta + \tilde{u}_0 + \sum_{i\geqslant 1} u_i \delta^{-i}, \tag{4.1}$$

which generates the Lax hierarchy (3.12) as the Δ-differential counterpart of the Kadomtsev–Petviashvili (KP) hierarchy.

For $(L)_{\geqslant 0} = \delta + \tilde{u}_0$, the first flow is given by

$$\begin{aligned} \frac{\mathrm{d}\tilde{u}_0}{\mathrm{d}t_1} &= \mu\Delta u_1 \\ \frac{\mathrm{d}u_i}{\mathrm{d}t_1} &= \sum_{k=0}^{i-1}(-1)^{k+1}u_{i-k}\sum_{j_1+j_2+\cdots+j_{k+1}=i}(E^{-j_{k+1}}\Delta E^{-j_k}\Delta\ldots E^{-j_2}\Delta E^{-j_1})\tilde{u}_0 \\ &\quad + \mu\Delta u_{i+1} + \Delta u_i + u_i\tilde{u}_0 \qquad \forall i > 0, \end{aligned} \tag{4.2}$$

where $j_\gamma > 0$ for all $\gamma \geqslant 1$.

For $(L^2)_{\geqslant 0} = \delta^2 + \xi\delta + \eta$, where

$$\xi := E\tilde{u}_0 + \tilde{u}_0 \qquad \eta := \Delta\tilde{u}_0 + \tilde{u}_0^2 + u_1 + Eu_1, \tag{4.3}$$

one calculates the second flow

$$\begin{aligned} \frac{\mathrm{d}u_0}{\mathrm{d}t_2} &= \mu\Delta(E+1)u_2 + \mu\Delta(\Delta u_1 + u_1\tilde{u}_0 + u_1E^{-1}\tilde{u}_0) \\ \frac{\mathrm{d}u_i}{\mathrm{d}t_2} &= \sum_{k=-1}^{i-1}(-1)^{k+2}u_{i-k}\sum_{j_1+j_2+\cdots+j_{k+2}=i+1}(E^{-j_{k+2}}\Delta E^{-j_{k+1}}\Delta\ldots E^{-j_2}\Delta E^{-j_1})\xi \\ &\quad + \sum_{k=0}^{i-1}(-1)^{k+1}u_{i-k}\sum_{j_1+j_2+\cdots+j_{k+1}=i}(E^{-j_{k+1}}\Delta E^{-j_k}\Delta\ldots E^{-j_2}\Delta E^{-j_1})\eta \\ &\quad + \Delta^2 u_i + (E\Delta + \Delta E)u_{i+1} + \mu\Delta(E+1)u_{i+2} + \xi(\Delta u_i + Eu_{i+1}) + \eta u_i, \end{aligned} \tag{4.4}$$

where $j_\gamma > 0$ for all $\gamma \geqslant 1$.

The simplest case in $(2+1)$ dimensions: we rewrite the first two members of the first flow by setting $\tilde{u}_0 = w$ and $t_1 = y$ and the first member of the second flow by setting $t_2 = t$. Since E and Δ do not commute, note that in the calculations up to the last step, we use $E - 1$ instead of $\mu\Delta$, to avoid confusion:

$$w_y = (E-1)u_1, \tag{4.5}$$

$$u_{1,y} = (E-1)u_2 + \Delta u_1 + u_1(1 - E^{-1})(w), \tag{4.6}$$

$$w_t = (E^2-1)u_2 + (E-1)(\Delta u_1 + u_1 w + u_1 E^{-1}(w)). \tag{4.7}$$

Applying $E + 1$ to (4.6) from the left yields

$$(E^2-1)u_2 = (E+1)u_{1,y} - (E+1)\Delta u_1 - (E-1)u_1(1 - E^{-1})w. \tag{4.8}$$

Applying $(E - 1)$ to (4.7) from the left and substituting (4.5) and (4.8) into the new derived equation we finally obtain the $(2+1)$-dimensional one-field system of the form

$$\mu\Delta w_t = (E+1)w_{yy} - 2\Delta w_y + 2\mu\Delta(ww_y), \tag{4.9}$$

which does not have a continuous counterpart. For the case of $\mathbb{T} = h\mathbb{Z}$, one can show that (4.9) is equivalent to the $(2+1)$-dimensional Toda lattice system.

The Δ-differential analog of one-field continuous KP equation is too complicated to write explicitly.

Remark 4.1. Here we want to illustrate the behavior of $\tilde{u}_0$ in all symmetries of the Δ-differential KP hierarchy. Let $(L^n)_{<0} = \sum_{i\geqslant 1} v_i^{(n)}\delta^{-i}$, then by the right-hand side of the Lax equation (3.12), we obtain the first members of all flows

$$\frac{\mathrm{d}\tilde{u}_0}{\mathrm{d}t_n} = \mu\Delta v_1^{(n)}. \tag{4.10}$$

Thus $\tilde{u}_0$ is time-independent for dense $x \in \mathbb{T}$ since $\mu = 0$. Hence when $\mathbb{T} = \mathbb{R}$, $\tilde{u}_0$ appears to be a constant.

In the $\mathbb{T} = \mathbb{R}$ case, or in the continuous limit of some special time scales, with $\tilde{u}_0 = 0$, the Lax operator (4.1) turns out to be a Laurent series of pseudo-differential operators

$$L = \partial + \sum_{i\geqslant 1} u_i\partial^{-i}. \tag{4.11}$$

Moreover, the first flow (4.2) turns out to be exactly the first flow of the KP system

$$\frac{\mathrm{d}u_i}{\mathrm{d}t_1} = u_{i,x}, \qquad \forall\, i \geqslant 1 \tag{4.12}$$

while the second flow (4.4) becomes exactly the second flow of the KP system

$$\frac{\mathrm{d}u_i}{\mathrm{d}t_2} = (u_i)_{2x} + 2(u_{i+1})_x + 2\sum_{k=1}^{i-1}(-1)^{k+1}\binom{i-1}{k}u_{i-k}(u_1)_{kx} \qquad \forall\, i \geqslant 1. \tag{4.13}$$

4.2. Δ-differential mKP, $k = 1$:

Consider the Lax operator of the form

$$L = \tilde{u}_{-1}\delta + \sum_{i\geqslant 0} u_i\delta^{-i} \tag{4.14}$$

which generates the Δ-differential counterpart of the modified Kadomstsev–Petviashvili (mKP) hierarchy.

Then, $(L)_{\geqslant 1} = \tilde{u}_{-1}\delta$ implies the first flow

$$\begin{aligned}
\frac{\mathrm{d}\tilde{u}_{-1}}{\mathrm{d}t_1} &= \mu\tilde{u}_{-1}\Delta u_0\\
\frac{\mathrm{d}u_i}{\mathrm{d}t_1} &= \sum_{k=-1}^{i-1}(-1)^{k+2}u_{i-k}\sum_{j_1+j_2+\cdots+j_{k+2}=i+1}(E^{-j_{k+2}}\Delta E^{-j_{k+1}}\Delta\ldots E^{-j_2}\Delta E^{-j_1})\tilde{u}_{-1}\\
&\quad + \tilde{u}_{-1}Eu_{i+1} + \tilde{u}_{-1}\Delta u_i \qquad \forall\, i \geqslant 0,
\end{aligned} \tag{4.15}$$

where $j_\gamma > 0$, $\gamma = 1, 2, \ldots, k+2$.

Next, for $(L^2)_{\geqslant 1} = \xi\delta^2 + \eta\delta$, where

$$\xi := \tilde{u}_{-1}E\tilde{u}_{-1}, \qquad \eta := \tilde{u}_{-1}\Delta\tilde{u}_{-1} + \tilde{u}_{-1}Eu_0 + u_0\tilde{u}_{-1}, \tag{4.16}$$

we have the second flow as follows:

$$\frac{d\tilde{u}_{-1}}{dt_2} = \xi(E\Delta u_0 + E^2(u_1)) + \mu\tilde{u}_{-1}\Delta u_0^2 - u_1 E^{-1}\xi - \tilde{u}_{-1}^2\Delta u_0$$

$$\begin{aligned}\frac{du_i}{dt_2} = &\sum_{k=-2}^{i-1}(-1)^{k+3}u_{i-k}\sum_{j_1+j_2+\cdots+j_{k+3}=i+2}(E^{-j_{k+3}}\Delta E^{-j_{k+2}}\Delta\ldots\Delta E^{-j_1})\xi\\ &+\sum_{k=-1}^{i-1}(-1)^{k+2}u_{i-k}\sum_{j_1+j_2+\cdots+j_{k+2}=i+1}(E^{-j_{k+2}}\Delta E^{-j_{k+1}}\Delta\ldots\Delta E^{-j_1})\eta\\ &+\xi(\Delta^2 u_i + (E\Delta + \Delta E)u_{i+1} + E^2 u_{i+2}) + \eta(\Delta u_i + Eu_{i+1}),\end{aligned} \tag{4.17}$$

where $i \geqslant 0$ and $j_\gamma > 0$ for all $\gamma \geqslant 1$.

Remark 4.2. Similarly in order to illustrate the behavior of $\tilde{u}_{-1}$ in all symmetries of the Δ-differential mKP hierarchy let us consider $(L^n)_{<1} = \sum_{i\geqslant 0} v_i^{(n)}\delta^{-i}$. Then we obtain the first members of all flows

$$\frac{d\tilde{u}_{-1}}{dt_n} = \mu\tilde{u}_{-1}\Delta v_0^{(n)}. \tag{4.18}$$

Thus $\tilde{u}_{-1}$ is time-independent for dense $x \in \mathbb{T}$. Hence when $\mathbb{T} = \mathbb{R}$, $\tilde{u}_{-1}$ appears to be a constant.

In the $\mathbb{T} = \mathbb{R}$ case, or in the continuous limit of some special time scales, with $\tilde{u}_{-1} = 1$, the Lax operator (4.14) turns out to be the pseudo-differential operator

$$L = \partial + \sum_{i\geqslant 0} u_i\partial^{-i}. \tag{4.19}$$

Furthermore, the system of equations (4.15) is exactly the first flow of the mKP system

$$\frac{du_i}{dt_1} = u_{i,x}, \qquad \forall i \geqslant 0, \tag{4.20}$$

while the second flow (4.17) turns out to be the second flow of the mKP system

$$\begin{aligned}\frac{du_i}{dt_2} = &(u_i)_{2x} + 2(u_{i+1})_x + 2u_0(u_i)_x + 2u_0u_{i+1}\\ &+2\sum_{k=0}^{i}(-1)^{k+1}\binom{i}{k}u_{i+1-k}(u_0)_{kx} \qquad \forall i \geqslant 0.\end{aligned} \tag{4.21}$$

5. Finite-field integrable systems on time scales

5.1. Δ-differential AKNS, $k = 0$:

Let the Lax operator (3.15) for $N = 1$ and $c_1 = 1$ be of the form

$$L = \delta + \tilde{u} + \psi\delta^{-1}\varphi. \tag{5.1}$$

The constraint (3.34) between fields, with $a_n = 0$, becomes

$$\tilde{u} = \mu\psi\varphi. \tag{5.2}$$

For $(L)_{\geqslant 0} = \delta + \tilde{u}$, one finds the first flow

$$\begin{aligned}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t_1} &= \mu\Delta(\psi E^{-1}\varphi),\\ \frac{\mathrm{d}\psi}{\mathrm{d}t_1} &= \tilde{u}\psi + \Delta\psi,\\ \frac{\mathrm{d}\varphi}{\mathrm{d}t_1} &= -\tilde{u}\varphi + \Delta E^{-1}\varphi.\end{aligned} \tag{5.3}$$

Eliminating field $\tilde{u}$ by (5.2) we have

$$\frac{\mathrm{d}\psi}{\mathrm{d}t_1} = \mu\psi^2\varphi + \Delta\psi, \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t_1} = -\mu\varphi^2\psi + \Delta E^{-1}\varphi. \tag{5.4}$$

Next we calculate $(L^2)_{\geqslant 0} = \delta^2 + \xi\delta + \eta$ where

$$\xi := (E+1)\tilde{u}, \qquad \eta := \Delta\tilde{u} + \tilde{u}^2 + \varphi E(\psi) + \psi E^{-1}(\varphi). \tag{5.5}$$

Thus, the second flow takes the form

$$\begin{aligned}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t_2} &= \mu\Delta[\Delta(\psi E^{-1}(\varphi)) + \psi E^{-1}(\tilde{u}\varphi) + \tilde{u}\psi E^{-1}\varphi] - \mu\Delta(E+1)\psi E^{-1}\Delta E^{-1}(\varphi)\\ \frac{\mathrm{d}\psi}{\mathrm{d}t_2} &= \psi\eta + \xi\Delta\psi + \Delta^2\psi\\ \frac{\mathrm{d}\varphi}{\mathrm{d}t_2} &= -\varphi\eta + \Delta E^{-1}(\xi\varphi) - (\Delta E^{-1})^2\varphi.\end{aligned} \tag{5.6}$$

By the use of the constraint (5.2), the second flow can be written as

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}t_2} &= \psi(\Delta\mu\psi\varphi + (\mu\psi\varphi)^2 + \varphi E(\psi) + \psi E^{-1}(\varphi)) + (E+1)\mu\psi\varphi\Delta\psi + \Delta^2\psi,\\ \frac{\mathrm{d}\varphi}{\mathrm{d}t_2} &= -\varphi(\Delta\mu\psi\varphi + (\mu\psi\varphi)^2 + \varphi E(\psi) + \psi E^{-1}(\varphi)) + \Delta E^{-1}(\varphi(E+1)\mu\psi\varphi) - (\Delta E^{-1})^2\varphi.\end{aligned} \tag{5.7}$$

In order to obtain the recursion operator one finds that for the Lax operator (5.1) the appropriate reminder (3.38) has the form

$$R = \Delta^{-1}(\mu^{-1}\tilde{u}_{t_n}) - \psi_{t_n}\delta^{-1}\varphi. \tag{5.8}$$

Then, (3.37) implies the following recursion formula as:

$$\begin{pmatrix}\tilde{u}\\ \psi\\ \varphi\end{pmatrix}_{t_{n+1}} = \begin{pmatrix}\tilde{u} - \mu^{-1} & \phi E & \psi E^{-1}\\ \psi + \psi\Delta^{-1}\mu^{-1} & \Delta + \tilde{u} + \psi\Delta^{-1}\varphi & \psi\Delta^{-1}\psi\\ -\varphi\Delta^{-1}\mu^{-1} & -\varphi E\Delta^{-1}\varphi & \tilde{u} - \Delta E^{-1} - \varphi E\Delta^{-1}\psi\end{pmatrix}\begin{pmatrix}\tilde{u}\\ \psi\\ \varphi\end{pmatrix}_{t_n} \tag{5.9}$$

valid for isolated points $x \in \mathbb{T}$, i.e. when $\mu \neq 0$. For dense points one must use its reduction by constraint (5.2)

$$\begin{pmatrix}\psi\\ \varphi\end{pmatrix}_{t_{n+1}} = \begin{pmatrix}\Delta + 2\mu\psi\varphi + 2\psi\Delta^{-1}\varphi & \mu\psi^2 + 2\psi\Delta^{-1}\psi\\ -\mu\varphi^2 - 2\varphi\Delta^{-1}\varphi & -\Delta E^{-1} - 2\varphi\Delta^{-1}\psi\end{pmatrix}\begin{pmatrix}\psi\\ \varphi\end{pmatrix}_{t_n}. \tag{5.10}$$

In the $\mathbb{T} = \mathbb{R}$ case, or in the continuous limit of some special time scales, with the choice $\tilde{u} = 0$, the Lax operator (5.1) takes the form $L = \partial + \psi\partial^{-1}\varphi$. Then, the continuous limits of (5.3) and (5.6), respectively, imply that the first flow is the translational symmetry

$$\frac{\mathrm{d}\psi}{\mathrm{d}t_1} = \psi_x \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t_1} = \varphi_x \tag{5.11}$$

and the first non-trivial equation from the hierarchy is the AKNS equation

$$\frac{d\psi}{dt_2} = \psi_{xx} + 2\psi^2\varphi, \qquad \frac{d\varphi}{dt_2} = -\varphi_{xx} - 2\varphi^2\psi. \tag{5.12}$$

For this special case the recursion formula (5.10) is of the following form:

$$\begin{pmatrix}\psi\\ \varphi\end{pmatrix}_{t_{n+1}} = \begin{pmatrix}\partial_x + 2\psi\partial_x^{-1}\varphi & 2\psi\partial_x^{-1}\psi\\ -2\varphi\partial_x^{-1}\varphi & -\partial_x - 2\varphi\partial_x^{-1}\psi\end{pmatrix}\begin{pmatrix}\psi\\ \varphi\end{pmatrix}_{t_n}. \tag{5.13}$$

5.2. *Δ-differential Kaup–Broer, $k = 1$:*

From the admissible finite-field restrictions (3.17), we consider the following simplest Lax operator:

$$L = \tilde{u}\delta + v + \delta^{-1}w. \tag{5.14}$$

The constraint (3.35), with $a_n = 1$, implies

$$\tilde{u} = 1 + \mu v - \mu^2 w. \tag{5.15}$$

Then, for $(L)_{\geqslant 1} = \tilde{u}\delta$, the first flow is given as

$$\begin{aligned}\frac{d\tilde{u}}{dt_1} &= \mu\tilde{u}\Delta v,\\ \frac{dv}{dt_1} &= \tilde{u}\Delta v + \mu\Delta E^{-1}(\tilde{u}w),\\ \frac{dw}{dt_1} &= \Delta E^{-1}(\tilde{u}w).\end{aligned} \tag{5.16}$$

By the constraint (5.15) one can rewrite the first flow as

$$\begin{aligned}\frac{dv}{dt_1} &= (1 + \mu v - \mu^2 w)\Delta v + \mu\Delta E^{-1}(w(1 + \mu v - \mu^2 w)),\\ \frac{dw}{dt_1} &= \Delta E^{-1}(w(1 + \mu v - \mu^2 w)).\end{aligned} \tag{5.17}$$

Next, we calculate $(L^2)_{\geqslant 1} = \xi\delta^2 + \eta\delta$, where

$$\xi := \tilde{u}E\tilde{u}, \qquad \eta := \tilde{u}\Delta\tilde{u} + \tilde{u}Ev + v\tilde{u}, \tag{5.18}$$

that yields the second flow

$$\begin{aligned}\frac{d\tilde{u}}{dt_2} &= \mu\tilde{u}\Delta(E^{-1} + 1)\tilde{u}w + \mu\tilde{u}\Delta v^2 + \mu\tilde{u}\Delta(\tilde{u}\Delta v),\\ \frac{dv}{dt_2} &= \xi(\Delta^2 v + \Delta w) + \mu\Delta E^{-1}(w\eta) + E^{-1}\Delta E^{-1}(w\xi) + \eta\Delta v,\\ \frac{dw}{dt_2} &= -\Delta E^{-1}\Delta E^{-1}(w\xi) + \Delta E^{-1}(w\eta).\end{aligned} \tag{5.19}$$

One can rewrite the above system reducing it by the constraint, but the final equation has a complicated form.

For the Lax operator (5.14) the appropriate reminder (3.39) is given by

$$R = \tilde{u}\Delta^{-1}(\mu\tilde{u})^{-1}\tilde{u}_{t_n}\delta - v_{t_n} - \Delta^{-1}w_{t_n}. \tag{5.20}$$

Hence, from (3.37) we have the following, valid when $\mu \neq 0$, recursion formula,

$$\begin{pmatrix}\tilde{u}\\ v\\ w\end{pmatrix}_{t_{n+1}} = \begin{pmatrix}R_{\tilde{u}\tilde{u}} & \tilde{u}E & \mu\tilde{u}\\ R_{v\tilde{u}} & v + \tilde{u}\Delta & (1 + E^{-1})\tilde{u}\\ R_{w\tilde{u}} & w & -\Delta E^{-1}\tilde{u} + v - \mu w\end{pmatrix}\begin{pmatrix}\tilde{u}\\ v\\ w\end{pmatrix}_{t_n}, \tag{5.21}$$

where

$$
\begin{aligned}
R_{\tilde{u}\tilde{u}} &= E(v) - \mu^{-1}\tilde{u} + \mu\tilde{u}\Delta(v)\Delta^{-1}(\mu\tilde{u})^{-1} \\
R_{v\tilde{u}} &= \Delta(v) + w + \tilde{u}\Delta(v)\Delta^{-1}(\mu\tilde{u})^{-1} + (1 - E^{-1})\tilde{u}w\Delta^{-1}(\mu\tilde{u})^{-1} \\
R_{w\tilde{u}} &= \Delta E^{-1}\tilde{u}w\Delta^{-1}(\mu\tilde{u})^{-1}.
\end{aligned}
\tag{5.22}
$$

Its reduction by the constraint (5.15) is

$$
\begin{pmatrix} v \\ w \end{pmatrix}_{t_{n+1}} = \begin{pmatrix} v + \tilde{u}\Delta + R_{v\tilde{u}}\mu & (1 + E^{-1})\tilde{u} - R_{v\tilde{u}}\mu^2 \\ w + R_{w\tilde{u}}\mu & -\Delta E^{-1}\tilde{u} + v - \mu w - R_{w\tilde{u}}\mu^2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_{t_n}, \tag{5.23}
$$

with $\tilde{u}$ given by (5.15).

In the case of $\mathbb{T} = \mathbb{R}$, or in the continuous limit of some special time scales, with the choice $\tilde{u} = 1$, the Lax operator (5.14) takes the form $L = \partial + v + \partial^{-1}w$. Then the similar continuous analog allows us to obtain the first flow

$$
\frac{\mathrm{d}v}{\mathrm{d}t_1} = v_x, \qquad \frac{\mathrm{d}w}{\mathrm{d}t_1} = w_x, \tag{5.24}
$$

and the first non-trivial equation from the hierarchy is the Kaup–Broer equation

$$
\frac{\mathrm{d}v}{\mathrm{d}t_2} = v_{2x} + 2w_x + 2vv_x, \qquad \frac{\mathrm{d}w}{\mathrm{d}t_2} = -w_{2x} + 2(vw)_x. \tag{5.25}
$$

For such special cases, the recursion formula (5.23) turns out to be

$$
\begin{pmatrix} v \\ w \end{pmatrix}_{t_{n+1}} = \begin{pmatrix} \partial_x + v + v_x\partial_x^{-1} & 2 \\ w + \partial_x w\partial_x^{-1} & -\partial_x + v \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_{t_n}. \tag{5.26}
$$

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References

[1] Błaszak M 1998 *Multi-Hamiltonian Theory of Dynamical Systems (Texts and Monograhps in Physics)* (Berlin: Springer) p 350

[2] Frenkel E 1996 Deformations of the KdV hierarchy and related soliton equations *Int. Math. Res. Not.* **1996** 55

[3] Adler M, Horozov E and van Moerbeke P 1998 The solution to the q-kdv equation *Phys. Lett.* A **242** 139

[4] Gürses M, Guseinov G Sh and Silindir B 2008 Integrable equations on time scales *J. Math. Phys.* **49** 072702

[5] Gelfand I M and Dickey L A 1976 Fractional powers of operators and Hamiltonian systems *Funct. Anal. Appl.* **10** 259–73

[6] Dickey L A 2003 *Soliton Equations and Hamiltonian Systems (Advanced Studies in Mathematical Physics* vol 26*)* (Singapore: World Scientific)

[7] Błaszak M, Gürses M, Silindir B and Szablikowski B M 2007 Integrable discrete systems on R and related dispersionless systems *Preprint* arXiv:0707.1084

[8] Reyman A G and Semenov-Tian-Shansky M A 1980 Family of Hamiltonian structures, hierarchy of Hamiltonians and reduction for matrix first order-differential operators *Funkz. Analys. Priloz.* **14** 77–8

[9] Semenov-Tian-Shansky M A 1983 What is a classical r-matrix? *Funct. Anal. Appl.* **17** 259

[10] Gürses M, Karasu A and Sokolov V V 1999 On construction of recursion operators from Lax representation *J. Math. Phys.* **40** 6473–90

[11] Bohner M and Peterson A 2001 *Dynamic Equations on Time Scales: An Introduction with Applications* (Boston, MA: Birkhauser)

[12] Bohner M and Peterson A (ed) 2003 *Advances in Dynamic Equations on Time Scales* (Boston, MA: Birkhauser)

[13] Aulbach B and Hilger S 1990 Linear dynamic process with inhomogeneous time scale *Nonlinear Dynamics and Quantum Dynamical Systems (Gaussing, 1990) (Math. Res.* vol 59*)* (Berlin: Akademie Verlag) pp 9–20

[14] Hilger S 1990 Analysis on measure chains—a unified approach to continuous and discrete calculus *Results Math.* **18** 18–56

[15] Atici F M and Guseinov G Sh 2002 On Greens functions and positive solutions for boundary value problems on time scales *J. Comp. Appl. Math.* **141** 75–99

[16] Konopelchenko B G and Oevel W 1993 An r-matrix approach to nonstandard classes of integrable equations *Publ. RIMS, Kyoto Univ.* **29** 581–666

[17] Oevel W and Strampp W 1993 Constrained KP hierarchy and bi-Hamiltonian structures *Commun. Math. Phys.* **157** 51

[18] Olver P J 2000 *Applications of Lie Groups to Differential Equations* (New York: Springer)

[19] Błaszak M 2001 On the construction of recursion operator and algebra of symmetries for field and lattice systems *Rep. Math. Phys.* **48** 27–38